Thermo-mechanical bending analysis of skew FGM laminated plates
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ABSTRACT
This study aims to investigate the thermo-mechanical bending analysis of skew FGM laminated plates in thermal environment. Formulation of laminated FGM plates have been performed with the help of Reddy’s higher order shear deformation theory. Material properties of plates are assumed to be temperature dependent and assumed to be vary in thickness direction as per power law distribution. Variational principle has been adopted to derive the governing equations of the problem. A C0 continuous finite element methodology has been used as solution methodology. Comparison and convergence study have been performed to check the efficiency and effectiveness of the present formulation. Various examples have been solved to demonstrate the effect of design parameter on the transverse deflection response of skew FGM laminated plates.

Keywords: Functionally graded materials, higher order shear deformation theory, skew plates, and finite element method.

1. INTRODUCTION
Composites are a class of the materials which is composed by integrating two or more materials macroscopically. These are heterogeneous materials, which can be tailored according to their intended applications [1]. Composites are widely used for the various applications. This is due to the fact that composites have high specific modulus, specific strength, and low thermal conductivity as compared to other traditional materials. However, traditional composite materials are unable to function properly in high thermal environment conditions, leads to delamination and debonding at the interfaces [2]. This need gave rise to a certain class of materials known as the functionally graded materials (FGMs).

In recent years the use of functionally graded materials (FGMs) have been increased in aerospace, defense and other industrial applications. FGMs are microscopic inhomogeneous materials in which mechanical properties are graded from one face to other face of structures. Like traditional composites FGMs are also made up of two or more types of materials generally metal and ceramic or combination of metals [3]. The use of ceramic makes it resistant to the temperature variations whereas the metal makes it ductile. Smooth gradation of FGM helps to achieve a single layer construction which helps to overcome stresses and

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sudden change in the material properties at layer interfaces. Along with this gradation helps to prevent the structures from debonding, delamination and other initial crack generation processes[4].

In certain cases, under high thermal loading the metal ceramic interface leads to some distortion, can initialize micro cracks and other initial failure phenomenon in the structure. In order to prevent this metallic and ceramic layer have been attached to FGM structure. Such structures are known as FGM laminates or sandwich FGM structures having homogeneous face sheets and FGM core [5]. Addition of homogeneous face sheets does not destroy the smooth gradation of material properties and helps to prevent the structure from initial failure phenomenon.

2. LITERATURE REVIEW

Sandwich FGM structures can be seen as a potential candidate for various thermomechanical applications. A large number of studies have been performed on sandwich FGM structures in last couple of years. Zenkour[6] studied the bending analysis of sandwich FGM plates having homogeneous face sheets while presenting a two-dimensional solution. The material properties of the constituent materials had been considered to be varying in thickness direction following the power law distribution. Later same author [7] presented another study on buckling and vibration behavior of functionally graded sandwich plates using sinusoidal shear deformation theory.

Li et al.[8] used 3D elasticity theory with Chebyshev polynomials to model the vibration response of two types of sandwich FGM plates. They used Ritz method to obtained the natural frequencies of the system. Xia and Shen [9] performed the large amplitude vibration analysis on initially post-buckled sandwich FGM structure under thermal environment. They employed HSDT with Von-Karman assumptions to formulate the plate kinematics. Improved perturbation technique had been used to solve the governing equations. Kiani et al. [10] performed the stability analysis on sandwich plates having FGM face sheets supported on elastic foundation. They employed FSDT for the formulation while using nonlinear strain displacement assumptions. Material properties of the plate are assumed to be temperature dependent and varying in plate thickness direction.

Taj and Chakrabarti [11] studies the static and dynamic behavior of skew FGM plates using TSDT. They used Mori-Tanaka scheme to calculate the effective material properties of the plates. Later they [12] analyzed the buckling response of skew FGM plates using an
efficient finite element methodology. Bessaim et al. [13] presented a higher shear and normal deformation theory and used it for the assessment of vibration and bending response of functionally graded sandwich plates having isotropic face sheets. Theory account the hyperbolic nature of shear deformation while satisfying the traction free condition at top and bottom face of plate. Houari et al.[14] performed thermoelastic analysis on functionally graded skew sandwich plates using a new shear and normal deformation theory having 5 number of unknowns. Theory account the thickness stretching affect while satisfying the stress-free boundary conditions.

Thai et al. [15] used a new FSDT to perform the bending, vibration and buckling behavior of sandwich FGM plates. They used Hamilton's principle to derive governing equations of plate. Results obtained from their theory are accurate as compared to traditional FSDT and comparable to other HSDT's. Taj et al. [16] performed bending analysis on the FGM skew sandwich plates with quadratic displacement variation in the kinematic field. Lagrangian element with 13 degrees of freedom per node was used to discretize the plate domain. Hamidi et al. [17] performed bending analysis on the functionally grade sandwich plates using a new sinusoidal five variable shear deformation theory. Theory accounts the effect of shear stretching while considering less number of unknowns than the standard sinusoidal theory. Bennoun et al. [18] gave a five variable refined plates theory to investigate the vibration behavior of sandwich FGM plates. Theory accounts the effect of thickness stretching and independent of shear correction factor.

3. MATHEMATICAL FORMULATION

3.1 Material Properties and Geometric Configuration

Consider a laminated FGM plate consists of ceramic, FGM, and metallic layers, respectively as shown Fig.1. The gradation of FGM layer follows the power law distribution. The effective material properties in the FGM zone can be written as [19],

$$P_{eff} = P_c V_c + P_m V_m$$ (1)

Where $V_c, V_m$ represents the volume fraction of the ceramic and metal, respectively.

Where, $V_c = \left( \frac{z_f - h_f}{h_f} \right)^n$, $h_f = h_1 + h_2$;

In order to consider the effect of thermal environment material properties are assumed to be temperature dependent[20],

$$P(t) = P_0(P_1 T^3 + 1 + P_2 T + P_3 T^2 + P_4 T^3),$$ (3)
Where, $P_0$, $P_1$, $P_2$ and $P_3$ represents the constants depend upon the type of constituent materials. A linear temperature distribution has been employed across the thickness of the plate which is written as,

$$T(z) = T_0 + \Delta T \left( \frac{z}{h} \right)$$  \hspace{1cm} (4)

3.2 Displacement field kinematics

The formulation is based on Reddy’s HSDT. A $C^0$ continuous displacement field is derived using the penalty approach [21].

$$
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} = \begin{bmatrix}
U_0 \\
V_0 \\
W_0
\end{bmatrix} + f_1(z) \begin{bmatrix}
\phi_1 \\
\phi_2 \\
0
\end{bmatrix} + f_2(z) \begin{bmatrix}
\theta_1 \\
\theta_2 \\
0
\end{bmatrix}
$$  \hspace{1cm} (5)

Von Karman strain assumptions have been employed to incorporate the geometric nonlinearity in the formulation. Strain vector considering the linear $\{\varepsilon_l\}$, nonlinear $\{\varepsilon_{nl}\}$ and thermal strain $\{\varepsilon_t\}$ is written as,

$$\varepsilon = \{\varepsilon_l\} + \{\varepsilon_{nl}\} - \{\varepsilon_t\}$$  \hspace{1cm} (6)

The constitutive relation is written as,

$$\{\sigma\} = [Q] \{\varepsilon\}$$  \hspace{1cm} (7)

Where, $\{\sigma\} \ [Q]$ and $\{\varepsilon\}$ represents the stress vector, reduced elastic coefficient matrix and strain vector, respectively.
3.3 Finite element implementation

A 9 noded isoparametric element has been used to discretize the domain of the plate. The strain energy equation after the finite element implementation can be written as [22],

\[
S = \sum_{\alpha=1}^{n} S^{\alpha} = \frac{1}{2} \sum_{\alpha=1}^{n} (\Gamma^{\alpha (T)} [K^{(\alpha)}] \Gamma^{\alpha})
\]  

(8)

Where, \([K^{\alpha}]\) represents the stiffness matrix for the element, \(\Gamma\) represents the displacement vector.

3.4 Skew boundary transformation

Boundaries of the plates are skewed and is not considered to be parallel to the global coordinates as shown in Fig. 2. Transformed displacement vector is written as [23],

\[\Gamma_i = T'_g \Gamma'_i,\]  

(9)

\[\text{Figure 2: Sandwich FGM plate with skew boundaries}\]

Where, \(T_g\) is transformation matrix consist of matrix containing sine and cosine terms with skew angle \((\psi)\). \(\Gamma_i\) and \(\Gamma'_i\) represents generalized and local displacement vectors at the respective ‘i’ node,

\[
\{\Gamma_i\} = \{U_0, V_0, W_0, \phi_0, \phi_y, \Theta_x, \Theta_y\}^T
\]

\[
\{\Gamma'_i\} = \{U'_0, V'_0, W'_0, \phi'_0, \phi'_y, \Theta'_x, \Theta'_y\}^T
\]

(10)

The final equilibrium equation for the bending analysis is written as,

\[ [K_{T_i}] [q] = F \]  

(11)

Where, \([K_{T_i}]\), \([q]\) and \(F\) represents the global transformed stiffness matrix, displacement vector and force vector, respectively.
4. RESULT AND DISCUSSIONS

This section have been divided into two parts. Initially convergence and comparison studies have been performed to prove the effectiveness of the present formulation. Later various numerical examples have been solved to demonstrate the effect of design parameters on non-dimensional deflection parameter. Table 1 and 2 respectively shows the material properties of various constituent materials used in the analysis.

Table 1. Material properties of constituent materials used in analysis

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
<th>E(N/m²)</th>
<th>υ</th>
<th>ρ(kg/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td></td>
<td>70 x 10⁹</td>
<td>0.30</td>
<td>2707</td>
</tr>
<tr>
<td>ZrO₂</td>
<td></td>
<td>151 x 10⁹</td>
<td>0.30</td>
<td>3000</td>
</tr>
<tr>
<td>Al₂O₃</td>
<td></td>
<td>105.7 x 10⁹</td>
<td>0.298</td>
<td>4429</td>
</tr>
</tbody>
</table>

Table 2. Temperature dependent material properties of constituent materials

<table>
<thead>
<tr>
<th>Material</th>
<th>Properties</th>
<th>P₀</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
<th>P(T=300K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZrO₂</td>
<td>E</td>
<td>0</td>
<td>244.27e9</td>
<td>-1.371e-3</td>
<td>1.214e-6</td>
<td>-3.68e-10</td>
</tr>
<tr>
<td></td>
<td>α</td>
<td>0</td>
<td>12.766 e9</td>
<td>-1.491e-3</td>
<td>1.006e-5</td>
<td>6.78e-11</td>
</tr>
<tr>
<td>Ti-6Al-4V</td>
<td>E</td>
<td>0</td>
<td>122.56 e9</td>
<td>-4.586e-4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>α</td>
<td>0</td>
<td>7.5788 e-6</td>
<td>6.638e-4</td>
<td>-3.147e-6</td>
<td>0</td>
</tr>
</tbody>
</table>

4.1 Convergence and comparison studies

In this section, two examples have been presented to demonstrate the reliability of the present formulation. Table 1 compares the bending response of the sandwich Al/Al₂O₃ FGM plate with Neves et al. [19]. They employed 3-D quasi higher order theory with Carrera’s unified formulation to derive the governing equations. Table 3 shows that results obtained with current formulation agrees well with literature. The plate is subjected to bi-sinusoidal transverse load, \( Q = Q \sin(\pi x / a) \sin(\pi y / b) \). Non-dimensional transverse displacement (\( w^* \)) have been considered to be: \( w^* = \frac{10h^3E}{a^4Q} w \)

Where, ‘\( w^* \)’, \( E_c \) and \( Q \) denotes the transverse deflection, elastic modulus of ceramic and load intensity respectively. The volume fraction index (n) has been considered as 1.

Table 3. Comparison of transverse displacement parameter (\( w^* \)) of sandwich FGM plate at various thickness ratios
Figure 3 shows the validation study for transverse displacement parameter \((w^*)\) of Al/ZrO\(_2\) skew FGM plates at various skew angles with Taj and Chakrabarti [11]. The volume fraction index \((n)\) of plates have been taken to be 2. Figure shows that results obtained with present formulation agrees well with literature.

\[
\begin{array}{cccc}
\text{Mesh size} & \text{Thickness ratio(a/h)} & 10 & 100 \\
2x2 & 0.8886 & 0.8217 \\
4x4 & 0.6300 & 0.5985 \\
5x5 & 0.6291 & 0.5996 \\
6x6 & 0.6279 & 0.5998 \\
\text{Ref. [19]} & 0.6305 & 0.6092 \\
\text{(%)} differ. & 0.4036 & 1.5448 \\
\end{array}
\]

4.2 Parametric study

In this section various examples have been shown to measure the effect of design parameters on non-dimensional transverse deflection of skew FGM laminated plate under bi-sinusoidal loading \(\bar{Q} = Q\sin(\pi x/a)\sin(\pi y/b)\). Non-dimensional transverse displacement \((w^*)\) have been considered to be:

\[
w^* = \frac{10h^2E_c}{a^4\bar{Q}w}
\]

**Example 1:** Figure 4 shows the deviation in non-dimensional transverse deflection of skew FGM laminated plates having simply supported edges with volume fraction index \((n)\). The thickness ratio and skew angle of the plate have been considered to be 10 and 100. Figure 4 shows that increase in volume fraction index leads to the decrease in non-dimensional transverse deflection. Whereas this behavior has been inverted in case of thickness ratio.
Example 2: Figure 5 shows the variation in the non-dimensional transverse deflection of Al-FGM-Al₂O₃ skew FGM laminated plates with skew angles. Plates are assumed to be under double sinusoidal loading conditions and supported at edges with simply supported boundary conditions. It can be observed from obtained results that non-dimensional transverse deflection reduces with decrease with skew angles. Whereas combined effect with thickness ratio also leads with further decrease in deflection parameter.

Example 3: Figure 6 shows the variation in the non-dimensional transverse deflection of Ti-6Al-4V/FGM/ZrO₂ skew FGM laminated plates with temperature difference (ΔT). The volume fraction index, thickness ratio and skew angles of plates have been considered to be 2, 10 and 100 respectively. It can be observed from calculated results that with the increase in
temperature difference deflection parameter increases. This is due to the reduction the stiffness of the sandwich FGM plate.

![Graph showing variation of transverse displacement parameter (w*) with temperature difference(ΔT)](image)

**Figure 5:** Variation of transverse displacement parameter (w*) with temperature difference(ΔT)

### 5. CONCLUSION

Bending analysis of skew functionally graded laminated plates have been investigated in thermal environment. Reddy’s higher order shear deformation theory have been used to model the displacement kinematics. Variational principle have been used to derive the plate governing equations. A C⁰ continuous 9 noded isoparametric element have been used to discretize the plate domain. Results obtained from present formulation shows good agreement with the literature. Whereas parametric study demonstrates that variation of design parameters such as skew angle (Ψ), thickness ratio(a/h), volume fraction index(n) and temperature difference(ΔT) directly influences the non-dimensional displacement parameter (w*) of skew FGM laminates.

### REFERENCES


