Time-Optimal Convergence of Fixed-Wing UAVs to a Circular Path in Steady Uniform Wind

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\textbf{ABSTRACT}
This paper explores the problem of generating time-optimal trajectories, starting from an arbitrary initial pose and converging to a circular path in two-dimensional space for a fixed-wing Unmanned Aerial Vehicle (UAV), constrained by a bounded turn radius. The strategy based on Dubins path has been presented to determine the optimal time path for the case where the UAV is initially located sufficiently far away from the circular path to converge. The analytical solution of the optimal time path in the absence of wind has been obtained geometrically. The work has also been extended to compute the optimal time path in the presence of steady uniform winds. Simulation results for both the cases in the absence and presence of constant winds have been produced to demonstrate the efficacy of this algorithm.

\textbf{Keywords}: Time-optimal trajectories, Unmanned Aerial Vehicle (UAV), Dubins path

\textbf{1. INTRODUCTION}
Path planning for UAVs has been an active field of research in recent years because of its wide range of applications in military and civilian sectors. Finding optimal time path is an important area for many applications like reconnaissance, surveillance which require UAVs to fly for long duration. The constraint involved in two-dimensional path planning for a fixed wing UAV is its minimum turn radius because of the limitations on its lateral acceleration.

The shortest path problem connecting two poses was first addressed by Dubins for a car moving forwards. A mathematical proof in [1] showed that the shortest path consists of line segments and arcs of circles of minimum turn radius. So, the shortest possible path that meets the maximum-curvature bound between two points with specific orientations in a plane is either a CLC or a CCC path, or a subset of them, where C represents circular arcs of minimum turn radius and L represents the straight line segment. The CLC path is formed by connecting two circular arcs by a line segment that is tangential to them, and the CCC path is formed by three consecutive circular arcs of minimum turn radius. So, the Dubins set, D, includes six paths D = (LSL, RSR, RSL, LSR, RLR, LRL), where left and right turn with minimum turn radius (r) are denoted by L and R, respectively. A variation of a Dubins path that allows backward motion is studied in Reeds and Shepp [2] using advanced calculus. Pontryagin’s Minimum Principle is used by Boissonnat et al. [3] to obtain the same results as in [1]. The lengths of the CLC paths for given poses are calculated by Shkel and Lumelsky [4]. Thomaschewski [5] solved a Dubins problem where terminal direction is not prescribed. Balluchi et al. [6] deduced the necessary conditions that

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the minimal-length paths must have to reach tangentially to the circular path using Pontryagin’s Maximum Principle.

One of the primary challenges with MAVs is flight in windy conditions. MAVs are expected to maneuver effectively in wind speeds which are 20%-60% of their airspeed. Dubins path is used to study the problem of finding an optimal path for an UAV in the presence of wind by McGee et al. [7]. A circular path-following method for MAVs in windy condition is developed by Nelson et al. [8] based on a vector field approach. However, the path followed by the MAV is not optimal. Kothari et al. [9] developed a guidance law that can track straight line paths, circular paths and combinations of both in the presence of wind. In [10], a strategy to determine the time-optimal trajectory for converging to a circular path in the presence of steady uniform winds is presented. However, in that solution, the vehicle is not reaching tangentially to the circular path which is required for following the circular path.

The main contribution of the present paper is to obtain a strategy which enables selecting the optimal time path for this problem in the absence of wind from an arbitrary initial pose without calculating the lengths of all the possible Dubins paths explicitly. For constant wind case, non-linear equations have been formulated to compute the time-optimal path. In this two dimensional problem, the terminal circular path is specified but not the final position. The case where the UAV is initially located sufficiently far away from the circular path is only considered in the present work. A comparison between the work in [9] and our approach has also been included in the current work to illustrate the strengths of the presented algorithm.

2. PROBLEM STATEMENT

The UAV is required to navigate along a path which would enable it to converge in a circular path in minimum time. For this purpose, the following kinematic model for the lateral dynamics of an UAV, with constant speed and altitude, is considered in a two dimensional space,

\[
\begin{align*}
\dot{x} &= V_a \cos \psi + V_w \cos \alpha \\
\dot{y} &= V_a \sin \psi + V_w \sin \alpha \\
\dot{\psi} &= \omega
\end{align*}
\]

where \(V_a\), \(V_w\) and \(V_g\) denotes airspeed, wind speed and ground speed of the UAV respectively, and \(\psi\) and \(\alpha\) denotes orientation angle of UAV airspeed and wind speed respectively. These variables are also explained in the Figure 1(a). The airspeed of UAV \((V_a)\) and wind speed \((V_w)\) are considered to be constant. The control input, \(\omega\) is constrained due to UAV’s bounded turn radius.

\[|\omega| \leq \frac{V_a}{r}\]

where \(r\) represents the minimum turn radius of the UAV.

The time-optimal trajectory is computed for an UAV from its known initial position and airspeed orientation to the given circular path. Without the loss of generality, it can be assumed that the center of the final circular path is at the origin of the coordinate system as shown in Figure 1(b). The radius of the circular path \((R)\) to be followed is assumed to be equal to or greater than the minimum turn radius \((r)\) of the vehicle, which is also necessary for tracking a circular path by the
The final X coordinate, $x_f$ and the final Y coordinate, $y_f$ are parameterised by the angle, $\theta_f$, as denoted in Figure 1(b), which is treated as a free parameter and are given by,

$$(x_f, y_f) \equiv (R \cos \theta_f, R \sin \theta_f)$$

3. OPTIMAL PATH GENERATION IN THE ABSENCE OF WIND

Let us first consider the case where the UAV has to converge in a circular path in the absence of wind. Without the loss of generality, the coordinate axes are considered to be oriented such that the UAV is initially located on -ve X axis. The vehicle with initial position $(x_s, 0)$ and orientation $(\psi_s$ with respect to X-axis) has to converge clockwise to a circular path of radius $R$. In this present work, the case where the vehicle is initially situated outside of the circle is considered. The same work can be extended for the case where the UAV is initially located inside the circular path. The vehicle is assumed to be initially located at a distance sufficiently away from the final circular path.

3.1. Possible Dubins Path

Time-optimal path in the absence of wind is same as the shortest length path with same constraints. The problem of finding the shortest smooth path for a given set of poses is solved in [4]. Paths possible for vehicle converging clockwise to a circular path from a sufficiently far away position are LSL and RSL. Figure 2 shows the circles corresponding to LSL and RSL path for a given initial position and heading of the vehicle. The coordinate of center of initial circular arc in the path is given by $(x_{cs}, y_{cs})$. The coordinate of center of final circular arc in the path is given by,$$(x_{cf}, y_{cf}) \equiv [(R + r) \cos \theta_f, (R + r) \sin \theta_f]$$

By sufficiently far away point, it is meant that the initial location of vehicle is such that there cannot be any optimal CCC or CC path for converging to the circular path. This condition can be ensured for an initial location of UAV which is greater than $(R+2r)$ [6]. The optimal path has been presented in this paper under this condition.
3.1.1 LSL Path

LSL path consists of two circular arcs of minimum turn radius corresponding to the left turns of the vehicle and a line segment which is tangential to both the circular arcs.

The coordinate of center of initial circular arc is given by,

\[(x_{cs}, y_{cs}) \equiv (x_s - r \sin \psi_s, r \cos \psi_s)\]

From Figure 3, \(\beta\) is the angle made by the line passing through \((x_{cs}, y_{cs})\) and \((x_{cf}, y_{cf})\) with positive x-axis.

\[\beta = \tan^{-1} \left( \frac{y_{cf} - y_{cs}}{x_{cf} - x_{cs}} \right)\]

Exit point of UAV, \((x_{ex}, y_{ex})\), from initial circle is given by,

\[(x_{ex}, y_{ex}) \equiv (x_{cs} + r \sin \beta, y_{cs} - r \cos \beta)\]

Entry point of UAV, \((x_{en}, y_{en})\), in final circle is given by,

\[(x_{en}, y_{en}) \equiv (x_{cf} + r \sin \beta, y_{cf} - r \cos \beta)\]

From Figure 4, Angle of rotation from initial point to exit point is given by,

\[\phi_s = (\beta - \psi_s + 2\pi) \quad \forall \quad \psi_s \in (0, 2\pi]\]

Angle of rotation from entry point to final point is given by,

\[\phi_f = (\theta_f - \beta - \pi/2)\]

Tangent Length is given by,

\[l_c = \sqrt{(x_{cs} - x_{cf})^2 + (y_{cs} - y_{cf})^2}\]
Total Path Length is given by,

\[ P = r(\phi_s + \phi_f) + l_c \]

For optimizing path length,

\[ \frac{dP}{d\phi_f} = 0 \]

It is found that,

\[ \frac{y_{en}}{x_{en}} = \frac{y_{ex}}{x_{ex}} \]

From this result, the following lemma can be written,

**Lemma 1** The line segment in the optimized LSL path belongs to a line passing through the center of the circular path to be converged.

### 3.1.2 RSL Path

RSL path consists of two circular arcs of minimum turn radius corresponding to the right and left turn of the vehicle respectively and a line segment which is tangential to both the circular arcs.
The coordinate of center of initial circular arc is given by,

\[(x_{cs}, y_{cs}) \equiv (x_s + r \sin \psi_s, -r \cos \psi_s)\]

From Figure 5, \(\beta\) is the angle made by the line passing through \((x_{cs}, y_{cs})\) and \((x_{cf}, y_{cf})\) with positive x-axis.

\[\beta = \tan^{-1} \left( \frac{y_{cf} - y_{cs}}{x_{cf} - x_{cs}} \right)\]

Exit point, \((x_{ex}, y_{ex})\), of UAV from initial circle is given by,

\[(x_{ex}, y_{ex}) \equiv [x_{cs} + r \cos(\beta + \alpha), y_{cs} + r \sin(\beta + \alpha)]\]

Entry point, \((x_{en}, y_{en})\), of UAV in final circle is given by,

\[(x_{en}, y_{en}) \equiv (x_{cf} - r \cos(\beta + \alpha), y_{cf} - r \sin(\beta + \alpha))\]

From Figure 6, Angle of rotation from initial point to exit point is given by,

\[\phi_s = (\pi/2 + \psi_s - \beta - \alpha) \quad \forall \quad \psi_s \in [0, 2\pi)\]

Angle of rotation from entry point to final point is given by,

\[\phi_f = (\theta_f - \beta - \alpha)\]

Tangent Length is given by,

\[l = \sqrt{(x_{ex} - x_{en})^2 + (y_{ex} - y_{en})^2}\]

Total Path Length is given by,

\[P = r(\phi_s + \phi_f) + l\]
For optimizing path length,
\[ \frac{dP}{d\theta_f} = 0 \]

It is found that,
\[ \frac{y_{ex}}{x_{ex}} = \frac{y_{en}}{x_{en}} \]

From this result, the following lemma can be written,

**Lemma 2** The line segment in the optimized RSL path belongs to a line passing through the center of the circular path to be converged.

### 3.2. Optimal RSL vs LSL Path

Using Lemma 1, optimal LSL path length is given by,
\[
P_{LSL} = r(3\pi/2 + \theta_f - \psi_s) + l_c
= r(3\pi/2 + \theta_f - \psi_s) + (x_s^2 - 2rx_s \sin \psi_s)^{\frac{1}{2}} - \left[R(R+2r)\right]^{\frac{1}{2}}
\]  

where \( \theta_f \) is given by,
\[
(R + r) \sin \theta_f = \frac{(x_s - r \sin \psi_s)r\left[R(R + 2r)\right]^{\frac{1}{2}} - (x_s^2 - 2rx_s \sin \psi_s + r^2)^{\frac{1}{2}}}{(x_s^2 - 2rx_s \sin \psi_s + r^2)} + \frac{r \cos \psi_s \left[r^2 + \left\{R(R + 2r)(x_s^2 - 2rx_s \sin \psi_s + r^2)\right\}^{\frac{1}{2}}\right]}{(x_s^2 - 2rx_s \sin \psi_s + r^2)}
\]

Using Lemma 2, optimal RSL path length is given by,
\[
P_{RSL} = r\left[\pi/2 + \theta_f + \psi_s - 2(\beta + \alpha)\right] + l
= r\left[\pi/2 + \theta_f + \psi_s - 2(\beta + \alpha)\right] + (x_s^2 + 2rx_s \sin \psi_s)^{\frac{1}{2}} - \left|R(R + 2r)\right]^{\frac{1}{2}}
\]
where $\theta_f - 2(\beta + \alpha)$ is given by,
\[
\sin(\theta_f - 2(\beta + \alpha)) = \frac{-(x_s + r\sin \psi_s)r\left\{R(R+2r)\right\}^{1/2} - (x_s^2 + 2rx_s\sin\psi_s + r^2)^{1/2}}{(R+r)(x_s^2 + 2rx_s\sin\psi_s + r^2)}
\]
\[
-\frac{r\cos \psi_s\left[r^2 + \{R(R+2r)(x_s^2 + 2rx_s\sin\psi_s + r^2)\}^{1/2}\right]}{(R+r)(x_s^2 + 2rx_s\sin\psi_s + r^2)}
\]

From above results, it can be seen that,
\[
(\theta_f)_{\text{LSL}} =\pi + [\theta_f - 2(\beta + \alpha)]_{\text{RSL}}
\]
\[
P_{\text{LSL}}(\psi_s = \chi) = P_{\text{RSL}}(\psi_s = 2\pi - \chi) \quad (6)
\]

It can also be found that,
\[
\frac{dP_{\text{LSL}}}{d\psi_s} \leq 0 \; ; \; \psi_s \in (0,2\pi]
\]
\[
& \frac{dP_{\text{LSL}}}{d\psi_s} < 0 \; ; \; \psi_s \in (0,2\pi) \quad (7)
\]

From eqns. (6) and (7),
\[
P_{\text{LSL}} > P_{\text{RSL}} \; \text{for} \; \psi_s \in (0,\pi),
\]
\[
P_{\text{LSL}} = P_{\text{RSL}} \; \text{for} \; \psi_s = \pi,
\]
\[
& P_{\text{LSL}} < P_{\text{RSL}} \; \text{for} \; \psi_s \in (\pi,2\pi)
\]

The above result is described in the following lemma,

**Lemma 3** The time-optimal path for converging to a circular path in the absence of wind is found to be RSL for initial orientation of vehicle’s airspeed in $(0, \pi]$ and LSL for initial orientation of vehicle’s airspeed in $[\pi, 2\pi)$ measured anticlockwise from the line joining the initial location of vehicle and the center of the circular path.

### 4. OPTIMAL PATH GENERATION IN STEADY UNIFORM WINDS

In this case, the vehicle has initial position $(x_s, y_s)$ and airspeed orientation ($\psi_s$ with respect to X-axis). Figure 7 depicts the problem situation in steady uniform winds. The final ground speed of UAV $(V_f)$ is given by,
\[
V_f \sin \theta_f = V_a \cos \psi_f + V_w \cos \alpha 
\]
\[
-V_f \cos \theta_f = V_a \sin \psi_f + V_w \sin \alpha
\]

where $\psi_f$ denotes final airspeed orientation. The optimal time airpath for this problem has to be Dubins path.

#### 4.1. Possible Dubins Airpaths

To minimize the time to reach $(x_f, y_f)$ from a sufficiently far position, the airpath has to be either RSL or LSL. Both the airpaths are analysed by dividing the path in three segments.
4.1.1 LSL Airpath

LSL airpath consists of three segments formulated in (a),(b) and (c) respectively in this section.

(a) For \(0 \leq t < t_1\), \(\dot{\psi} = \frac{V_a}{r}\psi\)
\[\psi(t) = \psi_s + \left(\frac{V_a}{r}\right)t\]

(b) For \(t_1 < t < (t_1 + t_2)\), \(\psi = 0\)
\[\psi(t) = \psi_1 = \psi_s + \left(\frac{V_a}{r}\right)t_1\]

(c) For \((t_1 + t_2) < t < (t_1 + t_2 + t_3)\), \(\psi = \frac{V_a}{r}\)
\[\psi(t) = \psi_f + \left(\frac{V_a}{r}\right)[t - (t_1 + t_2 + t_3)]\]

For optimizing total time \((t_1 + t_2 + t_3)\),
\[\frac{d(t_1 + t_2 + t_3)}{d\theta_f} = 0\]

The following non-linear equations [eqns. (10),(11),(12)] are formulated to compute \(t_1, t_2\) and \(t_3\) corresponding to optimal LSL airpath.

\[\begin{align*}
(V_a \cos \psi_1)t_2 + V_w \cos \alpha(t_1 + t_2 + t_3) &= x_f - r \sin \psi_f - x_{cs} \\
(V_a \sin \psi_1)t_2 + V_w \sin \alpha(t_1 + t_2 + t_3) &= y_f + r \cos \psi_f - y_{cs}
\end{align*}\]  
(10)

\[\frac{rV_f^3}{V_a R} [1 - \cos(\psi_f - \psi_1)] = [V_a \cos(\psi_f - \psi_1) + V_w \cos(\alpha - \psi_1)] [V_a + V_w \cos(\psi_f - \alpha)]\]  
(12)

where \((x_{cs}, y_{cs}) \equiv (x_s - r \sin \psi_s, y_s + r \cos \psi_s)\).
4.1.2 RSL Airpath

RSL airpath consists of three segments formulated in (a), (b) and (c) respectively in this section.

(a) For $0 \leq t < t_1$, $\dot{\psi} = -\frac{V_a}{r}$

$$\psi(t) = \psi_s - \left(\frac{V_a}{r}\right)t$$

(b) For $t_1 < t < (t_1 + t_2)$, $\dot{\psi} = 0$

$$\psi(t) = \psi_1 = \psi_s - \left(\frac{V_a}{r}\right)t_1$$

(c) For $(t_1 + t_2) < t < (t_1 + t_2 + t_3)$, $\dot{\psi} = \frac{V_a}{r}$

$$\psi(t) = \psi_f + \left(\frac{V_a}{r}\right)[t - (t_1 + t_2 + t_3)]$$

For optimizing total time $(t_1 + t_2 + t_3)$,

$$\frac{d(t_1 + t_2 + t_3)}{d\theta_f} = 0$$

The following non-linear equations [eqns. (13), (14), (15)] are formulated to compute $t_1$, $t_2$ and $t_3$ corresponding to optimal RSL airpath.

$$ (V_a \cos \psi_1) t_2 + V_w \cos \alpha (t_1 + t_2 + t_3) = x_f - r \sin \psi_f - x_{cs} + 2r \sin \psi_1$$

$$ (V_a \sin \psi_1) t_2 + V_w \sin \alpha (t_1 + t_2 + t_3) = y_f + r \cos \psi_f - y_{cs} - 2r \cos \psi_1$$

$$rV_f^3 \left[1 - \cos(\psi_f - \psi_1)\right] = [V_a \cos(\psi_f - \psi_1) + V_w \cos(\alpha - \psi_1)][V_a + V_w \cos(\psi_f - \alpha)]$$

where $(x_{cs}, y_{cs}) \equiv (x_s + r \sin \psi_s, y_s - r \cos \psi_s)$.

These non-linear equations can be solved using numerical methods to obtain the time-optimal LSL and RSL airpaths in the presence of steady uniform winds. After computing the flight time for both the airpaths, the minimum time path can be selected.

5. SIMULATION RESULTS

The time-optimal paths in the absence and presence of steady uniform wind are simulated for different initial conditions and the results are compared with the paths generated in [9]. Comparisons between the flight time in the absence of wind and in the presence of steady uniform wind for different initial conditions are given in Table 1a and Table 1b respectively. All the parameters are the same as mentioned in [9] where the UAV airspeed and wind speed are considered to be 13 m/s and 4 m/s respectively. The wind speed’s orientation angle is considered to be 23°. The minimum turn radius and the radius of circular path are considered to be 39 m and 50 m respectively. Non-linear equations have been solved for the cases given in Table 1b using the nonlinear
Figure 8: Trajectories of UAV in the absence of wind

system solver ‘fsolve’ provided by MATLAB. The optimal air-paths are found out to be RSL for all the cases in Table 1b. The flight time for first three cases in Table 1b have been compared with [9] and for the fourth case the initial condition is shifted little far from (-100,-100), that is, (-115, -115) m for constructing air-path of CSC type. The time-optimal path generation when the UAV is initially placed at close distances is beyond the scope of the current work. The UAV trajectories for these initial conditions in the absence of wind and in the presence of steady wind are depicted in Figure 8 and Figure 9 respectively.

Table 1: Initial conditions and corresponding flight time (sec)

<table>
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<tr>
<th>Case</th>
<th>X (m)</th>
<th>Y (m)</th>
<th>Heading Angle</th>
<th>Flight Time ([9])</th>
<th>Optimal Flight Time</th>
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<td>100</td>
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<td>π</td>
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</table>

From Table 1, it can be inferred that a considerable amount of time can be saved in a longer mission by following the strategy mentioned in this paper in steady uniform winds.

6. CONCLUSIONS AND FUTURE WORK

The central idea developed in this paper is to direct an UAV to its optimal time path which would enable it to follow a given circular path. We have considered the case where the vehicle is initially positioned at a sufficiently far away point from the circular path. We compared this optimal path
with that of [9] and found better results in terms of flight time. This work can be extended in case the UAV is initially positioned closer to the circular path to be followed. The work can also be extended to follow the circular path in the presence of time-varying wind.

REFERENCES


